Lecture 18

Hollow Waveguides

18.1 Hollow Waveguides

Hollow waveguides are useful for high-power microwaves. Air has a higher breakdown voltage compared to most materials, and hence, it could be a good medium for propagating high power microwave. Also, hollow metallic waveguides are sufficiently shielded from the rest of the world so that interference from other sources is minimized. Furthermore, for radio astronomy, they can provide a low-noise system immune to interference. Air generally has less loss than materials, and loss is often the source of thermal noise. A low loss waveguide is also a low noise waveguide.[1]

Many waveguide problems can be solved in closed form. An example is the coaxial waveguide previously discussed. In addition, there are many other waveguide problems that have closed form solutions. Closed form solutions to Laplace and Helmholtz equations are obtained by the separation of variables method. The separation of variables method works only for separable coordinate systems. (There are 11 separable coordinates for Helmholtz equation, but 13 for Laplace equation.) Some examples of separable coordinate systems are cartesian, cylindrical, and spherical coordinates. But these three coordinates are about all we need to know for solving many engineering problems. For other than these three coordinates, complex special functions need to be defined for their solutions, which are hard to compute. Therefore, more complicated cases are now handled with numerical methods using computers.

When a waveguide has a center conductor or two conductors like a coaxial cable, it can support a TEM wave where both the electric field and the magnetic field are orthogonal to the direction of propagation. The uniform plane wave is an example of a TEM wave, for instance. However, when the waveguide is hollow or is filled completely with a homogeneous medium, without a center conductor, it cannot support a TEM mode as we shall prove next. Much of the materials of this lecture can be found in [31, 76, 85].

[1] There is a fluctuation dissipation theorem [103, 104] that says that when a system loses energy to the environment, it also receives the same amount of energy from the environment in order to conserve energy. Hence, a lossy system loses energy to its environment, but it receives energy back from the environment in terms of thermal noise.

18.1.1 Absence of TEM Mode in a Hollow Waveguide

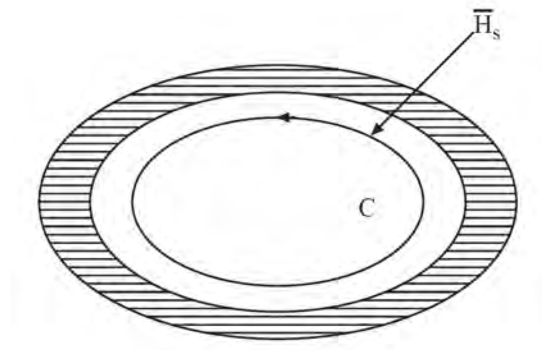

Figure 18.1: Absence of TEM mode in a hollow waveguide enclosed by a PEC wall. The magnetic field lines form a closed loop due to the absence of magnetic charges.

We would like to prove by contradiction (reductio ad absurdum) that a hollow waveguide as shown in Figure 18.1 (i.e. without a center conductor) cannot support a TEM mode as follows. If we assume that TEM mode does exist, then the magnetic field has to end on itself due to the absence of magnetic charges. It is clear that $\oint_C \mathbf{H}_s \cdot d\mathbf{l} \neq 0$ about any closed contour following the magnetic field lines. But Ampere's law states that the above is equal to

$$\oint_C \mathbf{H}_s \cdot d\mathbf{l} = j\omega\varepsilon \int_S \mathbf{E} \cdot d\mathbf{S} + \int_S \mathbf{J} \cdot d\mathbf{S} \tag{18.1.1}$$

Hence, this equation cannot be satisfied unless there are $E_z \neq 0$ component, or that $J_z \neq 0$ inside the waveguide. The right-hand side of the above cannot be entirely zero, or this implies that $E_z \neq 0$ unless a center conductor carrying a current $\mathbf{J}$ is there. This implies that a TEM mode in a hollow waveguide without a center conductor cannot exist.

Therefore, in a hollow waveguide filled with homogeneous medium, only TE_z (TE to z) or TM_z (TM to z) modes can exist like the case of a layered medium. For a TE_z wave (or TE wave), $E_z = 0$, $H_z \neq 0$ while for a TM_z wave (or TM wave), $H_z = 0$, $E_z \neq 0$. These classes of problems can be decomposed into two scalar problems like the layered medium case, by using the pilot potential method. However, when the hollow waveguide is filled with a center conductor, the TEM mode can exist in addition to TE and TM modes.

We will also study some closed form solutions to hollow waveguides, such as the rectangular waveguides. These closed form solutions offer us physical insight into the propagation of waves in a hollow waveguide. Another waveguide where closed form solutions can be obtained is the circular hollow waveguide. The solutions need to be sought in terms of Bessel functions. Another waveguide with closed form solutions is the elliptical waveguide. However, the solutions are too complicated to be considered.

18.1.2 TE Case ($E_z = 0,\ H_z \neq 0$)

In this case, the field inside the waveguide is TE to z or TE$_z$. To ensure such a TE field, we can write the **E** field as

$$\mathbf{E}(\mathbf{r}) = \nabla \times \hat{z}\Psi_h(\mathbf{r}) \tag{18.1.2}$$

Equation (18.1.2) will guarantee that $E_z = 0$ due to its construction. Here, $\Psi_h(\mathbf{r})$ is a scalar potential and $\hat{z}$ is called the pilot vector.[2]

The waveguide is assumed source free and filled with a lossless, homogeneous material. Eq. (18.1.2) also satisfies the source-free condition since $\nabla \cdot \mathbf{E} = 0$. And hence, from Maxwell's equations that

$$\nabla \times \mathbf{E} = j\omega\mu\mathbf{H} \tag{18.1.3}$$
$$\nabla \times \mathbf{H} = -j\omega\varepsilon\mathbf{E} \tag{18.1.4}$$

it can be shown that

$$\nabla \times \nabla \times \mathbf{E} - \omega^2\mu\varepsilon\mathbf{E} = 0 \tag{18.1.5}$$

Furthermore, using the appropriate vector identiy, such as the back-of-the-cab formula, it can be shown that the electric field $\mathbf{E}(\mathbf{r})$ satisfies the following Helmholtz wave equation, or partial differential equation that

$$(\nabla^2 + \beta^2)\mathbf{E}(\mathbf{r}) = 0 \tag{18.1.6}$$

where $\beta^2 = \omega^2\mu\varepsilon$. Substituting (18.1.2) into (18.1.6), we get

$$(\nabla^2 + \beta^2)\nabla \times \hat{z}\Psi_h(\mathbf{r}) = 0 \tag{18.1.7}$$

In the above, we assume that $\nabla^2\nabla \times \hat{z}\Psi = \nabla \times \hat{z}(\nabla^2\Psi)$, or that these operators commute.[3] Then it follows that

$$\nabla \times \hat{z}(\nabla^2 + \beta^2)\Psi_h(\mathbf{r}) = 0 \tag{18.1.8}$$

Thus, if $\Psi_h(\mathbf{r})$ satisfies the following Helmholtz wave equation or partial differential equation

$$(\nabla^2 + \beta^2)\Psi_h(\mathbf{r}) = 0 \tag{18.1.9}$$

then (18.1.8) is satisfied, and so is (18.1.6). Hence, the **E** field constructed with (18.1.2) satisfies Maxwell's equations, if $\Psi_h(\mathbf{r})$ satisfies (18.1.9).

[2]It "pilots" the field so that it is transverse to z.

[3]This is a mathematical parlance, and a commutator is defined to be $[A, B] = AB - BA$ for two operators A and B. If these two operators commute, then $[A, B] = 0$.

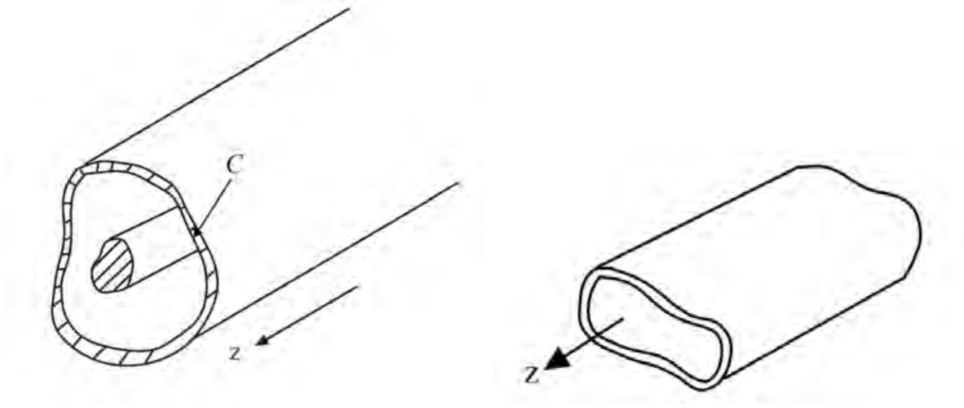

Figure 18.2: A hollow metallic waveguide with a center conductor (left), and without a center conductor (right).

Next, we look at the boundary condition for $\Psi_h(\mathbf{r})$ which is derivable from the boundary condition for $\mathbf{E}$. The boundary condition for $\mathbf{E}$ is that $\hat{n} \times \mathbf{E} = 0$ on C, the PEC wall of the waveguide. But from (18.1.2), using the back-of-the-cab (BOTC) formula,

$$\hat{n} \times \mathbf{E} = \hat{n} \times (\nabla \times \hat{z}\Psi_h) = -\hat{n} \cdot \nabla\Psi_h = 0 \tag{18.1.10}$$

In applying the BOTC formula, one has to be mindful that ∇ operates on a function to its right, and the function Ψ_h should be placed to the right of the ∇ operator.

In the above $\hat{n} \cdot \nabla = \hat{n} \cdot \nabla_s$ where $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$ since $\hat{n}$ has no z component. The boundary condition (18.1.10) then becomes

$$\hat{n} \cdot \nabla_s \Psi_h = \partial_n \Psi_h = 0 \text{ on } C \tag{18.1.11}$$

where C is the waveguide wall. The above is also known as the homogeneous Neumann boundary condition.

Furthermore, in a waveguide, just as in a transmission line case, we are looking for traveling solutions of the form $\exp(\mp j\beta_z z)$ for (18.1.9), or that

$$\Psi_h(\mathbf{r}) = \Psi_{hs}(\mathbf{r}_s)e^{\mp j\beta_z z} \tag{18.1.12}$$

where $\mathbf{r}_s = \hat{x}x+\hat{y}y$, or in short, $\Psi_{hs}(\mathbf{r}_s) = \Psi_{hs}(x, y)$. Thus, $\partial_n\Psi_h = 0$ implies that $\partial_n\Psi_{hs} = 0$. With this assumption, $\frac{\partial^2}{\partial z^2} \to -{\beta_z}^2$, and (18.1.9) becomes even simpler, namely that,

$$({\nabla_s}^2 + \beta^2 - {\beta_z}^2)\Psi_{hs}(\mathbf{r}_s) = ({\nabla_s}^2 + \beta_s^2)\Psi_{hs}(\mathbf{r}_s) = 0\ , \quad \partial_n\Psi_{hs}(\mathbf{r}_s) = 0, \text{ on } C \tag{18.1.13}$$

where $\nabla_s^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $\beta_s^2 = \beta^2 - \beta_z^2$. The above is a boundary value problem for a 2D waveguide problem. The above 2D wave equation is also called the reduced wave equation.

18.1.3 TM Case ($E_z \neq 0$, $H_z = 0$)

Repeating similar treatment for TM waves, the TM magnetic field is then

$$\mathbf{H} = \nabla \times \hat{z}\Psi_e(\mathbf{r}) \tag{18.1.14}$$

where

$$(\nabla^2 + \beta^2)\Psi_e(\mathbf{r}) = 0 \tag{18.1.15}$$

We need to derive the boundary condition for $\Psi_e(\mathbf{r})$ when we know that $\hat{n} \times \mathbf{E} = 0$ on the waveguide wall. To this end, we find the corresponding $\mathbf{E}$ field by taking the curl of the magnetic field in (18.1.14), and thus the $\mathbf{E}$ field is proportional to

$$\mathbf{E} \sim \nabla \times \nabla \times \hat{z}\Psi_e(\mathbf{r}) = \nabla\nabla \cdot (\hat{z}\Psi_e) - \nabla^2 \hat{z}\Psi_e = \nabla \frac{\partial}{\partial z}\Psi_e + \hat{z}\beta^2\Psi_e \tag{18.1.16}$$

where we have used the BOTC formula to simplify the above. Taking the z component of the above, we get

$$E_z \sim \frac{\partial^2}{\partial z^2}\Psi_e + \beta^2\Psi_e \tag{18.1.17}$$

Assuming that we have a propagating mode inside the waveguide so that

$$\Psi_e \sim e^{\mp j\beta_z z} \tag{18.1.18}$$

then in (18.1.17), $\partial^2/\partial z^2 \rightarrow -\beta_z^2$, and

$$E_z \sim (\beta^2 - \beta_z^2)\Psi_e \tag{18.1.19}$$

Therefore, if

$$\Psi_e(\mathbf{r}) = 0 \text{ on } C, \tag{18.1.20}$$

where C is the waveguide wall, then,

$$E_z(\mathbf{r}) = 0 \text{ on } C \tag{18.1.21}$$

Equation (18.1.19) is also called the homogeneous Dirichlet boundary condition. One can further show from (18.1.16) that the homogeneous Dirichlet boundary condition also implies that the other components of tangential $\mathbf{E}$ are zero, namely $\hat{n} \times \mathbf{E} = 0$ on the waveguide wall C.

Next, we assume that

$$\Psi_e(\mathbf{r}) = \Psi_{es}(\mathbf{r}_s)e^{\mp j\beta_z z} \tag{18.1.22}$$

This will allow us to replace $\partial^2/(\partial z)^2 = -\beta_z^2$. Thus, with some manipulation, the boundary value problem related to equation (18.1.15) reduces to a simpler 2D problem, i.e.,

$$({\nabla_s}^2 + \beta_s^2)\Psi_{es}(\mathbf{r}_s) = 0 \tag{18.1.23}$$

with the homogeneous Dirichlet boundary condition that

$$\Psi_{es}(\mathbf{r}_s) = 0,\ \mathbf{r}_s \text{ on } C \tag{18.1.24}$$

To illustrate the above theory, we can solve some simple waveguides problems.

18.2 Rectangular Waveguides

Rectangular waveguides are among the simplest waveguides to analyze because closed form solutions exist in cartesian coordinates. One can imagine traveling waves in the xy directions bouncing off the walls of the waveguide causing standing waves to exist inside the waveguide.

As shall be shown, it turns out that not all electromagnetic waves can be guided by a hollow waveguide. Only when the wavelength is short enough, or the frequency is high enough that an electromagnetic wave can be guided by a waveguide. When a waveguide mode cannot propagate in a waveguide, that mode is known to be cut-off. The concept of cut-off for hollow waveguide is quite different from that of a dielectric waveguide we have studied previously.

18.2.1 TE Modes (H Mode or $H_z \neq 0$ Mode)

For this mode, the scalar potential $\Psi_{hs}(\mathbf{r}_s)$ satisfies

$$(\nabla_s{}^2 + \beta_s{}^2)\Psi_{hs}(\mathbf{r}_s) = 0, \qquad \frac{\partial}{\partial n}\Psi_{hs}(\mathbf{r}_s) = 0 \quad \text{on } C \tag{18.2.1}$$

where $\beta_s{}^2 = \beta^2 - \beta_z{}^2$. A viable solution using separation of variables[4] for $\Psi_{hs}(x, y)$ is then

$$\Psi_{hs}(x, y) = A\cos(\beta_x x)\cos(\beta_y y) \tag{18.2.2}$$

where $\beta_x{}^2 + \beta_y^2 = \beta_s^2$. One can see that the above is the representation of standing waves in the xy directions. It is quite clear that $\Psi_{hs}(x, y)$ satisfies equation (18.2.1). Furthermore, cosine functions, rather than sine functions are chosen with the hindsight that the above satisfies the homogenous Neumann boundary condition at $x = 0$, and $y = 0$ surfaces.

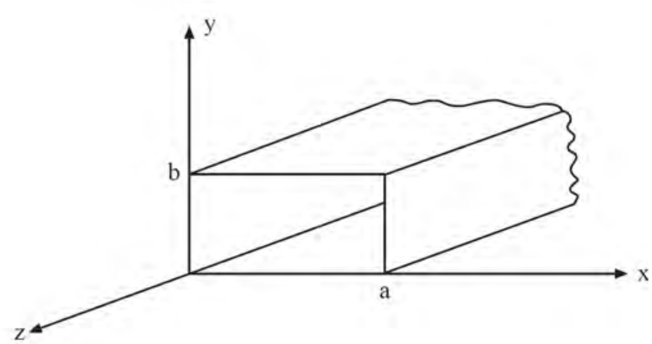

Figure 18.3: The schematic of a rectangular waveguide. By convention, the length of the longer side is usually named a.

[4]For those who are not familiar with this topic, please consult p. 385 of Kong [31].

To further satisfy the boundary condition at $x = a$, and $y = b$ surfaces, it is necessary that the boundary condition for eq. (18.1.11) is satisfied or that

$$\partial_x \Psi_{hs}(x, y)\big|_{x=a} \sim \sin(\beta_x a)\cos(\beta_y y) = 0, \tag{18.2.3}$$

$$\partial_y \Psi_{hs}(x, y)\big|_{y=b} \sim \cos(\beta_x x)\sin(\beta_y b) = 0, \tag{18.2.4}$$

The above puts constraints on β_x and β_y, implying that $\beta_x a = m\pi$, $\beta_y b = n\pi$ where m and n are integers. Hence (18.2.2) becomes

$$\Psi_{hs}(x, y) = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{18.2.5}$$

where

$$\beta_x^2 + \beta_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \beta_s^2 = \beta^2 - {\beta_z}^2 \tag{18.2.6}$$

Clearly, (18.2.5) satisfies the requisite homogeneous Neumann boundary condition at the entire waveguide wall.

At this point, it is prudent to stop and ponder on what we have done. Equation (18.2.1) is homomorphic to a matrix eigenvalue problem

$$\overline{\mathbf{A}} \cdot \mathbf{x}_i = \lambda_i \mathbf{x}_i \tag{18.2.7}$$

where $\mathbf{x}_i$ is the eigenvector and λ_i is the eigenvalue. Therefore, β_s^2 is actually an eigenvalue, and $\Psi_{hs}(\mathbf{r}_s)$ is an eigenfunction (or an eigenmode), which is analogous to an eigenvector. Here, the eigenvalue β_s^2 is indexed by m, n, so is the eigenfunction in (18.2.5). The corresponding eigenmode is also called the TE_{mn} mode.

The above condition on β_s^2 is also known as the guidance condition for the modes in the waveguide. Furthermore, from (18.2.6),

$$\beta_z = \sqrt{\beta^2 - \beta_s^2} = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{18.2.8}$$

And from (18.2.8), when the frequency is low enough, then

$$\beta_s^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 > \beta^2 = \omega^2\mu\varepsilon \tag{18.2.9}$$

and β_z becomes pure imaginary and the mode cannot propagate or becomes evanescent in the z direction.[5] For fixed m and n, the frequency at which the above happens is called the cutoff frequency of the TE_{mn} mode of the waveguide. It is given by

$$\omega_{mn,c} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{18.2.10}$$

[5] We have seen this happening in a plasma medium earlier and also in total internal reflection.

When $\omega < \omega_{mn,c}$, the TE_{mn} mode is evanescent and cannot propagate inside the waveguide. A corresponding cutoff wavelength is then

$$\lambda_{mn,c} = \frac{2}{[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2]^{1/2}} \tag{18.2.11}$$

So when $\lambda > \lambda_{mn,c}$, the mode cannot propagate inside the waveguide.

When $m = n = 0$, then $\Psi_h(\mathbf{r}) = \Psi_{hs}(x, y)\exp(\mp j\beta_z z)$ is a function independent of x and y. Then $\mathbf{E}(\mathbf{r}) = \nabla \times \hat{z}\Psi_h(\mathbf{r}) = \nabla_s \times \hat{z}\Psi_h(\mathbf{r}) = 0$. It turns out the only way for $H_z \neq 0$ is for $\mathbf{H}(\mathbf{r}) = \hat{z}H_0$ which is a static field in the waveguide. This is not a very interesting mode, and thus TE_{00} propagating mode is assumed not to exist and not useful. So the TE_{mn} modes cannot have both $m = n = 0$. As such, the TE_{10} mode, when $a > b$, is the mode with the lowest cutoff frequency or longest cutoff wavelength. Only when the frequency is above this cutoff frequency and the wavelength is shorter than this cutoff wavelength, the TE_{10} mode can propagate.

For the TE_{10} mode, for the mode to propagate, from (18.2.11), it is needed that

$$\lambda < \lambda_{10,c} = 2a \tag{18.2.12}$$

The above has the nice physical meaning that the wavelength has to be smaller than $2a$ in order for the mode to fit into the waveguide. As a mnemonic, we can think that photons have "sizes", corresponding to its wavelength. Only when its wavelength is small enough can the photons go into (or be guided by) the waveguide. The TE_{10} mode, when $a > b$, is also the mode with the lowest cutoff frequency or longest cutoff wavelength.

It is seen with the above analysis, when the wavelength is short enough, or frequency is high enough, many modes can be guided. Each of these modes has a different group and phase velocity. But for most applications, a single guided mode only is desirable. Hence, the knowledge of the cutoff frequencies of the fundamental mode (the mode with the lowest cutoff frequency) and the next higher mode is important. This allows one to pick a frequency window within which only a single mode can propagate in the waveguide.

It is to be noted that when a mode is cutoff, the field is evanescent, and there is no real power flow down the waveguide: Only reactive power is conveyed by such a mode.